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G C S E M a t h e m a t i c s

Higher Tier

Challenge Questions

*Ten questions pitched above standard GCSE demand
100 marks · Answers provided*

Topics covered

1. Index laws
2. Changing the subject
3. Functions and inverses
4. Ratio
5. Straight lines and gradients
6. Circles and tangents
7. Sectors and segments
8. Capture and recapture
9. Inequalities
10. Circle theorems

1

[11 marks]

- (a) Simplify fully, giving your answer in the form ka^mb^n , where $b > 0$. (3)

$$(16a^8b^{-4})^{\frac{3}{4}} \div (2a^2b^{-1})^2$$

- (b) Solve the equation (4)

$$\frac{9^x \times 27^2}{3^{x+1}} = 81^{x-1}$$

- (c) Solve the equation (4)

$$2^{2x} - 5(2^x) + 4 = 0$$

2

[9 marks]

- (a) Make x the subject of (4)

$$y = \sqrt{\frac{3x+2}{x-5}}$$

- (b) Make x the subject of (5)

$$y = \frac{\sqrt{x}+1}{\sqrt{x}-2}$$

3

[10 marks]

The functions f and g are defined as follows.

$$f(x) = \frac{3x-1}{x+2}, \quad x \neq -2$$

$$g(x) = x^2 + 4$$

- (a) Find $f^{-1}(x)$, stating clearly the value that x cannot take. (3)
- (b) Solve $fg(x) = 2$. (4)
- (c) Explain why the equation $f(x) = 3$ has no solution. (3)

4

[9 marks]

- (a) For three numbers a , b and c , $a : b = 3 : 4$ and $b : c = 6 : 7$. Given that $a + c = 69$, find b . (4)
- (b) A bag contains only red and blue counters, in the ratio $5 : 3$. Twelve red counters are removed and eight blue counters are added. The ratio of red to blue is now $1 : 1$. Find the number of red counters originally in the bag. (5)

5

[11 marks]

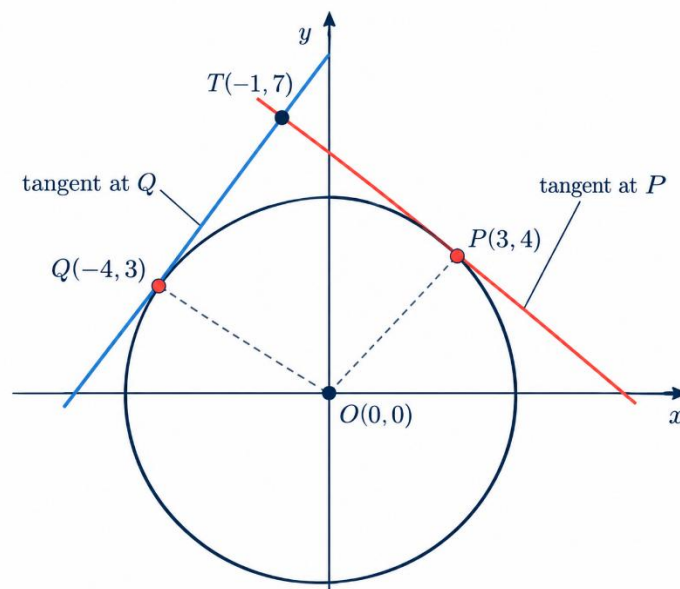
The points A and B have coordinates $(-2, 5)$ and $(6, 1)$.

- (a) Find the equation of the perpendicular bisector of AB , giving your answer in the form $y = mx + c$. (4)
- (b) This bisector crosses the axes at C and D . Find the exact area of triangle OCD , where O is the origin. (3)
- (c) The line L is parallel to AB and passes through $(0, 7)$. Find the exact coordinates of the point where L meets the perpendicular bisector. (4)

6

[11 marks]

The circle shown has equation $x^2 + y^2 = 25$. The points $P(3, 4)$ and $Q(-4, 3)$ lie on the circle.

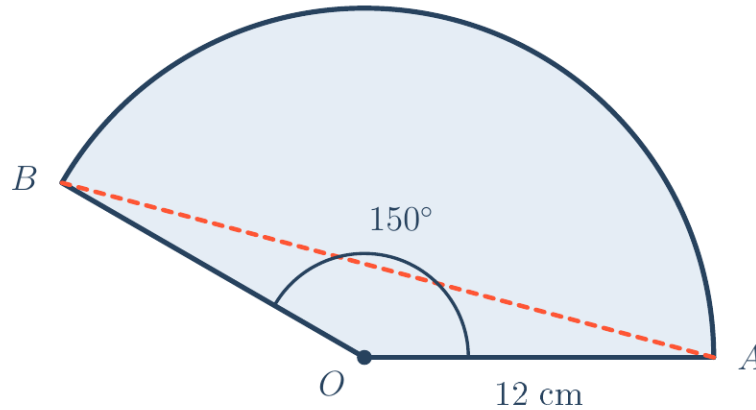


- (a) Find the equation of the tangent to the circle at P , giving your answer in the form $ax + by = c$. (3)
- (b) This tangent crosses the axes at A and B . Find the exact area of triangle OAB . (4)
- (c) The tangent at Q has equation $-4x + 3y = 25$. Find the coordinates of T , where the two tangents meet. (4)

7

[10 marks]

The diagram shows a sector OAB of a circle of radius 12 cm. The area of the sector is $60\pi \text{ cm}^2$.



- (a) Show that the angle of the sector is 150° . (3)
- (b) Find the exact perimeter of the sector. (3)
- (c) The chord AB is drawn. Find the exact area of the shaded segment between the chord and the arc. (4)

8

[9 marks]

A biologist is estimating the number of trout in a lake.

- (a) She catches 120 trout, marks them and returns them. Later she catches 80 trout, of which 12 are marked. Estimate the number of trout in the lake. (3)
- (b) In a second lake she marks 150 trout. A later sample of n trout contains 18 marked trout, giving an estimate of 1000 trout. Find n . (3)
- (c) State two assumptions that must be made for these estimates to be valid. (3)

9

[10 marks]

- (a) Solve the inequality (3)

$$2x^2 + 5x - 12 \leq 0$$

- (b) Solve the inequality (4)

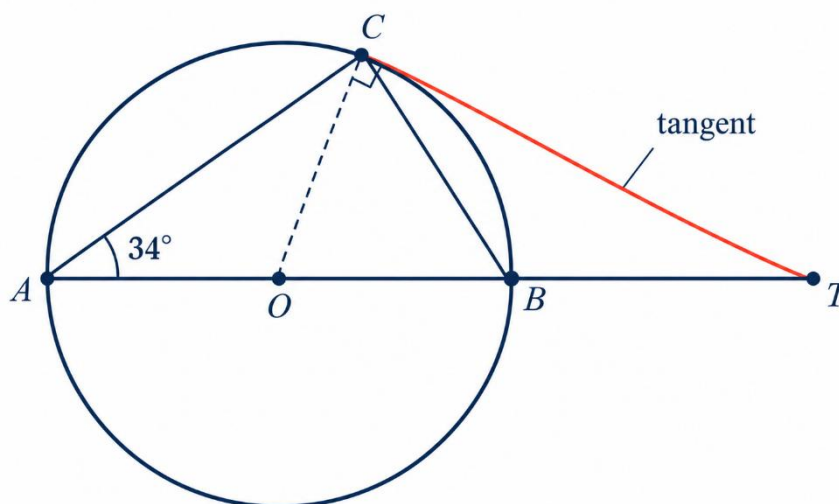
$$\frac{x - 4}{x + 3} \leq 0$$

- (c) Hence write down all the integer values of x that satisfy both inequalities. (3)

10

[10 marks]

In the diagram, A , B and C lie on a circle centre O . AB is a diameter. The tangent at C meets AB extended at T . Angle $CAB = 34^\circ$.



- (a) Write down the size of angle ACB , giving a reason. (2)
- (b) Find the size of angle ABC . (2)
- (c) Find the size of angle BCT , giving a reason. (3)
- (d) Find the size of angle BTC . (3)

Answers

1

(a)

$$(16a^8b^{-4})^{\frac{3}{4}} = 8a^6b^{-3}$$

$$(2a^2b^{-1})^2 = 4a^4b^{-2}$$

$$\frac{8a^6b^{-3}}{4a^4b^{-2}} = 2a^2b^{-1} = \frac{2a^2}{b}$$

(b)

$$3^{2x} \times 3^6 \div 3^{x+1} = 3^{4x-4}$$

$$3^{x+5} = 3^{4x-4} \Rightarrow x+5 = 4x-4$$

$$\mathbf{x = 3}$$

(c)

$$\text{Let } u = 2^x : u^2 - 5u + 4 = 0$$

$$(u-1)(u-4) = 0 \Rightarrow 2^x = 1 \text{ or } 2^x = 4$$

$$\mathbf{\underline{x = 0 \text{ or } x = 2}}$$

2

(a)

$$y^2(x-5) = 3x+2 \Rightarrow x(y^2-3) = 5y^2+2$$

$$x = \frac{5y^2+2}{y^2-3}$$

$$\mathbf{\underline{x = (5y^2 + 2) / (y^2 - 3)}}$$

(b)

$$y(\sqrt{x} - 2) = \sqrt{x} + 1 \Rightarrow \sqrt{x}(y - 1) = 2y + 1$$

$$x = \left(\frac{2y + 1}{y - 1} \right)^2$$

$$\mathbf{x = ((2y + 1) / (y - 1))^2}$$

3

(a)

$$y(x + 2) = 3x - 1 \Rightarrow x(y - 3) = -1 - 2y$$

$$f^{-1}(x) = \frac{2x + 1}{3 - x}, \quad x \neq 3$$

$$\mathbf{f^{-1}(x) = (2x + 1) / (3 - x), \quad x \neq 3}$$

(b)

$$fg(x) = \frac{3(x^2 + 4) - 1}{(x^2 + 4) + 2} = \frac{3x^2 + 11}{x^2 + 6}$$

$$3x^2 + 11 = 2x^2 + 12 \Rightarrow x^2 = 1$$

$$\mathbf{x = 1 \text{ or } x = -1}$$

(c) Rearranging $f(x) = 3$ gives $3x - 1 = 3x + 6$, which reduces to $-1 = 6$.

There is no value of x satisfying this, so $f(x) = 3$ has no solution. Equivalently, $y = 3$ is a horizontal asymptote of the graph.

No solution: $y = 3$ is never attained

4

(a)

$$a : b = 9 : 12 \quad b : c = 12 : 14 \quad a : b : c = 9 : 12 : 14$$

$$9k + 14k = 69 \Rightarrow k = 3$$

$$\mathbf{b = 36}$$

(b) Let the original numbers be $5x$ and $3x$.

$$\frac{5x - 12}{3x + 8} = 1 \Rightarrow 5x - 12 = 3x + 8$$

$$x = 10 \Rightarrow \text{red} = 50$$

50 red counters

5

(a)

$$M = (2, 3) \quad m_{AB} = \frac{1 - 5}{6 + 2} = -\frac{1}{2}$$

$$\text{perpendicular gradient} = 2 \Rightarrow y = 2x - 1$$

$$**y = 2x - 1**$$

(b)

$$C(0, -1) \quad D\left(\frac{1}{2}, 0\right) \quad \text{area} = \frac{1}{2} \times \frac{1}{2} \times 1 = \frac{1}{4}$$

$$**Area = 1/4**$$

(c)

$$2x - 1 = -\frac{1}{2}x + 7 \Rightarrow \frac{5}{2}x = 8$$

$$\left(\frac{16}{5}, \frac{27}{5}\right)$$

$$**(16/5, 27/5)**$$

6

(a)

$$m_{OP} = \frac{4}{3} \Rightarrow m_{\text{tangent}} = -\frac{3}{4}$$

$$3x + 4y = 25$$

$$**3x + 4y = 25**$$

(b)

$$A\left(\frac{25}{3}, 0\right) \quad B\left(0, \frac{25}{4}\right)$$

$$\text{area} = \frac{1}{2} \times \frac{25}{3} \times \frac{25}{4} = \frac{625}{24}$$

$$\mathbf{\underline{Area = 625/24}}$$

(c) Solving the two tangent equations simultaneously:

$$3x + 4y = 25 \quad -4x + 3y = 25$$

$$T(-1, 7)$$

$$\mathbf{\underline{T(-1, 7)}}$$

7

(a)

$$\frac{\theta}{360} \times \pi \times 12^2 = 60\pi \Rightarrow \theta = 150^\circ$$

$$\mathbf{\underline{\theta = 150^\circ}}$$

(b)

$$\text{arc} = \frac{150}{360} \times 2\pi \times 12 = 10\pi$$

$$\text{perimeter} = (10\pi + 24) \text{ cm}$$

$$\mathbf{\underline{Perimeter = (10\pi + 24) \text{ cm}}}$$

(c)

$$\text{triangle} = \frac{1}{2} \times 12^2 \times \sin 150^\circ = 36$$

$$\text{segment} = (60\pi - 36) \text{ cm}^2$$

$$\mathbf{\underline{Segment = (60\pi - 36) \text{ cm}^2}}$$

8

(a)

$$N = \frac{120 \times 80}{12} = 800$$

About 800 trout

(b)

$$\frac{150 \times n}{18} = 1000 \Rightarrow n = 120$$

n = 120

(c) Any two of the following.

The marked fish mix evenly back through the population before the second sample is taken.

No fish are born, die, enter or leave the lake between the two samples.

The marks do not come off, and marked fish are no more or less likely to be caught.

Two valid assumptions

9

(a)

$$(2x - 3)(x + 4) \leq 0 \Rightarrow -4 \leq x \leq \frac{3}{2}$$

$-4 \leq x \leq 3/2$

(b) Critical values are found where the numerator and denominator are zero. Note that $x = -3$ must be excluded.

$$\text{critical values } x = 4, x = -3 \Rightarrow -3 < x \leq 4$$

$-3 < x \leq 4$

(c) The overlap of the two solution sets is $-3 < x \leq 3/2$.

$$-3 < x \leq \frac{3}{2} \Rightarrow x = -2, -1, 0, 1$$

$x = -2, -1, 0, 1$

10

(a) The angle in a semicircle is a right angle.

Angle ACB = 90°

(b) Angles in triangle ABC sum to 180°, so $180 - 90 - 34$.

Angle ABC = 56°

- (c) By the alternate segment theorem, the angle between the tangent and chord CB equals the angle in the alternate segment, which is angle CAB .

Angle BCT = 34°

- (d) Triangle OBC is isosceles, so angle $OCB = 56^\circ$ and angle $BOC = 68^\circ$.

The radius meets the tangent at 90° , so in triangle OCT : $68 + 90 + \text{angle } OTC = 180$.

Angle BTC = 22°

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