



A Level Mathematics • Pure Mathematics

The Quotient Rule

Theory, Worked Examples and Graded Exercises

What this booklet contains

- The quotient rule stated in full, with both standard notations.
- A table of standard derivatives needed throughout.
- Three fully worked examples showing the expected layout.
- Ten graded questions, moving from routine practice to examination standard.
- Full worked solutions with method marks indicated.

Topic	Differentiation of a quotient of two functions
Level	A Level (Year 1 and Year 2 Pure)
Total marks	49
Suggested time	75 minutes

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1. The quotient rule

If y is formed by **dividing** one function of x by another, the derivative is found using the quotient rule. Unlike the product rule, the order of the two terms matters, because they are subtracted.

The quotient rule

$$y = \frac{u}{v} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

In function notation, if $f(x) = \frac{g(x)}{h(x)}$ then

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}$$

How to set the work out

1. Let u be the **numerator** and v be the **denominator**. Never the other way round.
2. Differentiate each one separately to get $\frac{du}{dx}$ and $\frac{dv}{dx}$.
3. Substitute into the rule, keeping the subtraction in the correct order.
4. Simplify the numerator only. Leave the denominator as v^2 in bracket form, since a factorised denominator is easier to work with and stationary points depend on the numerator alone.

Two things worth remembering

- **Say it out loud:** bottom times derivative of the top, minus top times derivative of the bottom, all over bottom squared.
- **Stationary points:** since $v^2 > 0$ wherever the function is defined, solving $\frac{dy}{dx} = 0$ only requires the numerator to be set equal to zero.

Standard derivatives used in this booklet

y	$\frac{dy}{dx}$	y	$\frac{dy}{dx}$
x^n	nx^{n-1}	$\sin x$	$\cos x$
$\cos x$	$-\sin x$	$\tan x$	$\sec^2 x$
e^{kx}	$k e^{kx}$	$\ln x$	$\frac{1}{x}$
$\sin kx$	$k \cos kx$	$\cos kx$	$-k \sin kx$

Common mistakes to avoid

- Writing the numerator the wrong way round as $u \frac{dv}{dx} - v \frac{du}{dx}$. This gives the derivative with the wrong sign.
- Forgetting to square the denominator.
- Expanding v^2 unnecessarily, which usually makes the answer harder to simplify.
- Using the quotient rule when simplifying first is quicker. For example $y = \frac{x^4 + 3x^2}{x^2}$ should be written as $y = x^2 + 3$ before differentiating.

2. Worked examples

Example 1 • A linear quotient

Differentiate $y = \frac{x+2}{x-4}$.

Step 1: choose u and v

$$u = x + 2, \quad v = x - 4$$

Step 2: differentiate each one

$$\frac{du}{dx} = 1, \quad \frac{dv}{dx} = 1$$

Step 3: apply the quotient rule

$$\frac{dy}{dx} = \frac{(x-4)(1) - (x+2)(1)}{(x-4)^2}$$

Step 4: simplify the numerator only

$$\frac{dy}{dx} = \frac{x-4-x-2}{(x-4)^2}$$

$$\frac{dy}{dx} = -\frac{6}{(x-4)^2}$$

Note that the derivative is negative for all $x \neq 4$, so the curve is decreasing everywhere it is defined.

Example 2 • A quadratic numerator

Given $y = \frac{2x^2-1}{x+3}$, find $\frac{dy}{dx}$.

Step 1: choose u and v , then differentiate

$$u = 2x^2 - 1, \quad \frac{du}{dx} = 4x, \quad v = x + 3, \quad \frac{dv}{dx} = 1$$

Step 2: apply the quotient rule

$$\frac{dy}{dx} = \frac{(x + 3)(4x) - (2x^2 - 1)(1)}{(x + 3)^2}$$

Step 3: simplify the numerator

$$\frac{dy}{dx} = \frac{4x^2 + 12x - 2x^2 + 1}{(x + 3)^2}$$

$$\frac{dy}{dx} = \frac{2x^2 + 12x + 1}{(x + 3)^2}$$

Example 3 • Finding stationary points

The curve C has equation $y = \frac{x}{x^2 + 1}$. Find the coordinates of the stationary points of C .

Step 1: choose u and v , then differentiate

$$u = x, \quad \frac{du}{dx} = 1, \quad v = x^2 + 1, \quad \frac{dv}{dx} = 2x$$

Step 2: apply the quotient rule

$$\frac{dy}{dx} = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{x^2 + 1 - 2x^2}{(x^2 + 1)^2}$$

$$\frac{dy}{dx} = \frac{1 - x^2}{(x^2 + 1)^2}$$

Step 3: set the numerator equal to zero

The denominator is always positive, so

$$1 - x^2 = 0 \implies x = 1 \text{ or } x = -1$$

Step 4: find the corresponding y values

$$x = 1 \implies y = \frac{1}{2}, \quad x = -1 \implies y = -\frac{1}{2}$$

$$\left(1, \frac{1}{2}\right) \text{ and } \left(-1, -\frac{1}{2}\right)$$

3. Exercise: the quotient rule

Answer every question. Show all method, including a clear statement of u , v , $\frac{du}{dx}$ and $\frac{dv}{dx}$. Give exact answers throughout. Total: **49 marks**.

Section A • Warm up (Questions 1 to 3)

Q1. Differentiate with respect to x , giving your answer in its simplest form:

$$y = \frac{x}{x + 1}$$

[3 marks]

Q2. Find $\frac{dy}{dx}$ when

$$y = \frac{2x + 1}{3x - 2}$$

[3 marks]

Q3. Differentiate with respect to x :

$$y = \frac{x^2 + 1}{x - 3}$$

[4 marks]

Section B • Building fluency (Questions 4 to 7)

Q4. A curve has equation $y = \frac{\sin x}{x}$, for $x \neq 0$.

(a) Find $\frac{dy}{dx}$.

[3]

(b) Hence find the exact gradient of the curve at the point where $x = \pi$.

[1]

[4 marks]

Q5. Show that if $y = \frac{e^{2x}}{x + 1}$ then

$$\frac{dy}{dx} = \frac{e^{2x}(2x + 1)}{(x + 1)^2}$$

[4 marks]

Q6. Find $\frac{dy}{dx}$, simplifying the numerator fully, when

$$y = \frac{3x - 1}{x^2 + 4}$$

[5 marks]

Q7. Starting from $\tan x = \frac{\sin x}{\cos x}$:

(a) Use the quotient rule to prove that $\frac{d}{dx}(\tan x) = \sec^2 x$. [4]

(b) Hence write down the gradient of the curve $y = \tan x$ at the point where $x = \frac{\pi}{4}$. [1]

[5 marks]

Section C • Stretch and examination standard (Questions 8 to 10)

Q8. The curve C has equation $y = \frac{\ln x}{x^2}$, for $x > 0$.

(a) Show that $\frac{dy}{dx} = \frac{1 - 2 \ln x}{x^3}$. [3]

(b) Find the exact coordinates of the stationary point of C . [2]

(c) Determine whether this stationary point is a maximum or a minimum, giving a reason. [1]

[6 marks]

Q9. The curve C has equation $y = \frac{x^2 + 3}{x - 1}$, for $x \neq 1$.

(a) Show that $\frac{dy}{dx} = \frac{(x - 3)(x + 1)}{(x - 1)^2}$. [4]

(b) Find the coordinates of the two stationary points of C . [2]

(c) Determine the nature of each stationary point, giving a reason. [1]

[7 marks]

Q10. The curve C has equation

$$y = \frac{1 - \cos x}{1 + \cos x}, \quad 0 < x < \pi$$

- (a) Show that $\frac{dy}{dx} = \frac{2 \sin x}{(1 + \cos x)^2}$. [4]
- (b) Explain why C has no stationary points for $0 < x < \pi$. [1]
- (c) Find the equation of the tangent to C at the point where $x = \frac{\pi}{2}$, giving your answer in the form $y = mx + c$ with m and c exact. [3]

[8 marks]

4. Full worked solutions

Q1. $y = \frac{x}{x+1}$. $u = x, v = x+1$, so $\frac{du}{dx} = 1$ and $\frac{dv}{dx} = 1$.

$$\frac{dy}{dx} = \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{x+1-x}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{1}{(x+1)^2}$$

Q2. $y = \frac{2x+1}{3x-2}$. $u = 2x+1, v = 3x-2$, so $\frac{du}{dx} = 2$ and $\frac{dv}{dx} = 3$.

$$\frac{dy}{dx} = \frac{(3x-2)(2) - (2x+1)(3)}{(3x-2)^2} = \frac{6x-4-6x-3}{(3x-2)^2}$$

$$\frac{dy}{dx} = -\frac{7}{(3x-2)^2}$$

Q3. $y = \frac{x^2+1}{x-3}$. $u = x^2+1, v = x-3$, so $\frac{du}{dx} = 2x$ and $\frac{dv}{dx} = 1$.

$$\frac{dy}{dx} = \frac{(x-3)(2x) - (x^2+1)(1)}{(x-3)^2} = \frac{2x^2-6x-x^2-1}{(x-3)^2}$$

$$\frac{dy}{dx} = \frac{x^2-6x-1}{(x-3)^2}$$

Q4. $y = \frac{\sin x}{x}$.

(a)

$u = \sin x, v = x$, so $\frac{du}{dx} = \cos x$ and $\frac{dv}{dx} = 1$.

$$\frac{dy}{dx} = \frac{x \cos x - \sin x \cdot 1}{x^2}$$

$$\frac{dy}{dx} = \frac{x \cos x - \sin x}{x^2}$$

(b)

At $x = \pi$: $\cos \pi = -1$ and $\sin \pi = 0$, so

$$\frac{dy}{dx} = \frac{\pi(-1) - 0}{\pi^2} = -\frac{1}{\pi}$$

Q5. $y = \frac{e^{2x}}{x+1}$. $u = e^{2x}, v = x+1$, so $\frac{du}{dx} = 2e^{2x}$ and $\frac{dv}{dx} = 1$.

$$\frac{dy}{dx} = \frac{(x+1)(2e^{2x}) - e^{2x}(1)}{(x+1)^2}$$

Take out the common factor e^{2x} in the numerator:

$$\frac{dy}{dx} = \frac{e^{2x}[2(x+1) - 1]}{(x+1)^2} = \frac{e^{2x}(2x+2-1)}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{e^{2x}(2x+1)}{(x+1)^2} \quad \text{as required}$$

Q6. $y = \frac{3x-1}{x^2+4}$. $u = 3x-1$, $v = x^2+4$, so $\frac{du}{dx} = 3$ and $\frac{dv}{dx} = 2x$.

$$\frac{dy}{dx} = \frac{(x^2+4)(3) - (3x-1)(2x)}{(x^2+4)^2}$$

$$= \frac{3x^2 + 12 - 6x^2 + 2x}{(x^2+4)^2}$$

$$\frac{dy}{dx} = \frac{12 + 2x - 3x^2}{(x^2+4)^2}$$

Q7. $y = \tan x = \frac{\sin x}{\cos x}$.

(a)

$u = \sin x$, $v = \cos x$, so $\frac{du}{dx} = \cos x$ and $\frac{dv}{dx} = -\sin x$.

$$\frac{dy}{dx} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

Using the identity $\sin^2 x + \cos^2 x = 1$:

$$\frac{dy}{dx} = \frac{1}{\cos^2 x}$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \text{as required}$$

(b)

At $x = \frac{\pi}{4}$: $\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$, so $\sec^2 \frac{\pi}{4} = 2$. The gradient is 2.

Q8. $y = \frac{\ln x}{x^2}.$

(a)

$u = \ln x, v = x^2$, so $\frac{du}{dx} = \frac{1}{x}$ and $\frac{dv}{dx} = 2x$.

$$\frac{dy}{dx} = \frac{x^2 \cdot \frac{1}{x} - \ln x \cdot 2x}{(x^2)^2} = \frac{x - 2x \ln x}{x^4}$$

Cancel a factor of x from the numerator and denominator:

$$\frac{dy}{dx} = \frac{1 - 2 \ln x}{x^3} \quad \text{as required}$$

(b)

For $x > 0$ the denominator is never zero, so set the numerator equal to zero:

$$1 - 2 \ln x = 0 \quad \Rightarrow \quad \ln x = \frac{1}{2} \quad \Rightarrow \quad x = e^{\frac{1}{2}} = \sqrt{e}$$

$$y = \frac{\ln \sqrt{e}}{(\sqrt{e})^2} = \frac{\frac{1}{2}}{e}$$

$$\left(\sqrt{e}, \frac{1}{2e} \right)$$

(c)

Consider the sign of $\frac{dy}{dx} = \frac{1 - 2 \ln x}{x^3}$ either side of $x = \sqrt{e} \approx 1.65$.

At $x = 1$: $\frac{dy}{dx} = \frac{1 - 0}{1} = 1 > 0$. At $x = e$: $\frac{dy}{dx} = \frac{1 - 2}{e^3} < 0$.

The gradient changes from positive to negative, so the stationary point is a **maximum**.

Q9. $y = \frac{x^2 + 3}{x - 1}.$

(a)

$$u = x^2 + 3, v = x - 1, \text{ so } \frac{du}{dx} = 2x \text{ and } \frac{dv}{dx} = 1.$$

$$\frac{dy}{dx} = \frac{(x-1)(2x) - (x^2+3)(1)}{(x-1)^2} = \frac{2x^2 - 2x - x^2 - 3}{(x-1)^2}$$

$$\frac{dy}{dx} = \frac{x^2 - 2x - 3}{(x-1)^2}$$

Factorising the numerator:

$$\frac{dy}{dx} = \frac{(x-3)(x+1)}{(x-1)^2} \text{ as required}$$

(b)

Since $(x-1)^2 > 0$ for $x \neq 1$, set the numerator equal to zero:

$$x = 3 \Rightarrow y = \frac{9+3}{2} = 6, \quad x = -1 \Rightarrow y = \frac{1+3}{-2} = -2$$

$$(3, 6) \text{ and } (-1, -2)$$

(c)

The sign of $\frac{dy}{dx}$ is the sign of $(x-3)(x+1)$.

At $x = 2$: negative. At $x = 4$: positive. So the gradient goes from negative to positive and $(3, 6)$ is a **minimum**.

At $x = -2$: positive. At $x = 0$: negative. So the gradient goes from positive to negative and $(-1, -2)$ is a **maximum**.

Q10. $y = \frac{1 - \cos x}{1 + \cos x}.$

(a)

$$u = 1 - \cos x, v = 1 + \cos x, \text{ so } \frac{du}{dx} = \sin x \text{ and } \frac{dv}{dx} = -\sin x.$$

$$\frac{dy}{dx} = \frac{(1 + \cos x)(\sin x) - (1 - \cos x)(-\sin x)}{(1 + \cos x)^2}$$

$$= \frac{\sin x + \sin x \cos x + \sin x - \sin x \cos x}{(1 + \cos x)^2}$$

$$\frac{dy}{dx} = \frac{2 \sin x}{(1 + \cos x)^2} \quad \text{as required}$$

(b)

For $0 < x < \pi$, $\sin x > 0$, so the numerator $2 \sin x$ is never zero. The denominator is positive, so $\frac{dy}{dx} > 0$ throughout the interval and C has no stationary points. The curve is increasing on this interval.

(c)

At $x = \frac{\pi}{2}$: $\cos \frac{\pi}{2} = 0$ and $\sin \frac{\pi}{2} = 1$, so

$$y = \frac{1 - 0}{1 + 0} = 1, \quad \frac{dy}{dx} = \frac{2(1)}{(1 + 0)^2} = 2$$

Using $y - y_1 = m(x - x_1)$ at the point $\left(\frac{\pi}{2}, 1\right)$ with $m = 2$:

$$y - 1 = 2 \left(x - \frac{\pi}{2} \right)$$

$$y = 2x - \pi + 1$$

Next steps

Once the quotient rule is secure, practise deciding *which* rule to use before you start writing. Many examination questions can be done either by the quotient rule or by rewriting the expression as a product with a negative index, and choosing well saves a great deal of algebra. Then move on to mixed questions that combine the product, quotient and chain rules with connected rates of change.